

Quantum Effects of an Extra Compact Dimension on the Wave Function of the Universe

Guowen Peng¹ , Hongya Liu^{1,2} , and Qiuyang Zhang¹

Abstract

We extend the direct quantum approach of the standard FRW cosmology from $4D$ to $5D$ and obtain a Hamiltonian formulation for a wave-like $5D$ FRW cosmology. Using a late-time approximation we isolate out a y -part from the full wave function of the $5D$ Universe. Then we find that the compactness of the fifth dimension y yields a quantized spectrum for the momentum P_5 along the fifth dimension, and we show that the whole space-part of the wave function of the $5D$ Universe satisfies a two-dimensional Schrödinger equation.

Key words: Quantum cosmology; Hamiltonian formulaiton.

¹ Department of Physics, Dalian University of Technology, Dalian, 116024, People's Republic of China.

² Corresponding author, E-mail: hylu@dlut.edu.cn.

1 Introduction

Quantum cosmology has received extensive studies (Wheeler et al. 1964; Novello et al. 1995; Anchordoqui et al. 2000). Recently, a direct quantum approach was used to derive Hamiltonian formulation of the standard Friedmann-Robertson-Walker (FRW) cosmological models (Elbaz et al. 1997; Novello et al. 1996). The method is based on assumptions of validity of the classical FRW equations and a transformation from a pair of physical variables (ρ, θ) (the density of matter and the inverse of the Hubble radius), which describe the dynamics of a spatially homogeneous and isotropic perfect fluid, to another pair of canonical variables (q, p) (roughly the radius of the Universe and its rate of change). In this way, a canonical description of quantum FRW cosmology was obtained. In this paper, we wish to extend this procedure to a $5D$ FRW cosmology and discuss quantum effects of the fifth compact dimension on the wave function of the Universe.

The arrangement of this paper is as follows. In section 2, we study a class of $5D$ wave-like cosmological solutions and derive Hamiltonian formulation of the $5D$ FRW cosmology. In section 3, we use a late-time approximation to separate the $5D$ wave function of the Universe and isolate out the part of the fifth dimension. In section 4, we study quantization of the wave function due to the compactness of the fifth dimension. Section 5 is a conclusion.

2 Hamiltonian Formulation of 5D Cosmology

We consider a FRW type $5D$ metric

$$dS^2 = -B^2(t, y)dt^2 + A^2(t, y)\left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right) + C^2(t, y)dy^2, \quad (2.1)$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2 \theta d\varphi^2$ and the coordinates $x^A = (t, r\theta\varphi, y)$ (Here and throughout this paper, lowercase Greek letters run 0, 1, 2, 3 and uppercase Latin letters run 0, 1, 2, 3, 4, 5). Using

the $4D$ part of the $5D$ metric (2.1) we can calculate all the non-vanishing components of the $4D$ Einstein tensor ${}^{(4)}G_{\alpha\beta}$, which are as follows:

$${}^{(4)}G_0^0 = 3\frac{\dot{A}^2}{A^2B^2} + \frac{3k}{A^2} , \quad (2.2)$$

$${}^{(4)}G_1^1 = {}^{(4)}G_2^2 = {}^{(4)}G_3^3 = -2\frac{\ddot{A}}{AB^2} - \frac{\dot{A}^2}{A^2B^2} - \frac{k}{A^2} + 2\frac{\dot{A}\dot{B}}{AB^3} . \quad (2.3)$$

where a dot denotes partial derivative with respect to time t .

It is known that solutions which are empty in $5D$ may have matter in $4D$ (Wesson 1999; Overduin & Wesson 1997). Therefore, one can define an induced $4D$ energy-momentum tensor $T_{\alpha\beta}$ as

$$T_{\alpha\beta} \equiv {}^{(4)}G_{\alpha\beta} . \quad (2.4)$$

It is found that this $T_{\alpha\beta}$ can take the form of a perfect fluid,

$$T_{\alpha\beta} = (\rho + P)u_\alpha u_\beta + Pg_{\alpha\beta} , \quad (2.5)$$

where ρ and P are the energy density and pressure of the induced matter, respectively.

Now we let the equation of state being

$$P = \gamma\rho , \quad (2.6)$$

where $\gamma = 0$ and $\gamma = \frac{1}{3}$ represent matter-dominated and radiation-dominated eras, respectively. Then substituting (2.2), (2.3), (2.5) and (2.6) in (2.4), we obtain the well-known Friedmann equation

$$3\frac{\dot{A}^2}{A^2} = B^2\rho - \frac{3kB^2}{A^2} , \quad (2.7)$$

and the Raychaudhuri one

$$3\frac{\ddot{A}}{A} - 3\frac{\dot{A}\dot{B}}{AB} + \frac{B^2}{2}(1 + 3\gamma)\rho = 0 . \quad (2.8)$$

The conservation law for the energy-momentum $T_{\alpha\beta}$ gives us another equation

$$\dot{\rho} + 3(1 + \gamma)\frac{\dot{A}}{A}\rho = 0 . \quad (2.9)$$

An exact 5D cosmological solution satisfying the empty 5D equations $R_{AB} = 0$ is (Liu & Wesson 1994)

$$dS^2 = -B^2(u)dt^2 + A^2(u)(dr^2 + r^2d\Omega^2) + B^2(u)dy^2 , \quad (2.10)$$

where k is chosen to be zero,

$$\begin{aligned} A(u) &= (hu)^{\frac{1}{2+3\gamma}} , \\ B(u) &= (hu)^{-\frac{1+3\gamma}{2(2+3\gamma)}} , \end{aligned} \quad (2.11)$$

and $u \equiv t - y$. The energy density ρ and pressure P are

$$\rho = \frac{3h^2}{(2+3\gamma)^2}A^{-3(1+\gamma)}, \quad P = \gamma\rho . \quad (2.12)$$

This is a wave-like cosmological solution which can be interpreted as a shock wave propagating along the fifth dimension (Wesson et al. 2000).

Now let us return to equations (2.7)-(2.9). There are three basic equations from which we want to derive the Hamiltonian formulation. Note that since $A = A(u)$ and $u = t - y$, we have $\dot{A} = \partial A / \partial t = dA / du$. Therefore, by using a transformation

$$\tilde{u} = \int B(u)du , \quad (2.13)$$

we can reduce equations (2.7)-(2.9) to

$$\frac{3}{A^2} \overset{*}{A}^2 = \rho , \quad (2.14)$$

$$\frac{3}{A} \overset{**}{A} + \frac{1+3\gamma}{2}\rho = 0 , \quad (2.15)$$

$$\dot{\rho}^* + 3(1 + \gamma) \frac{\dot{A}}{A} \rho = 0 , \quad (2.16)$$

where an asterisk denotes derivative with respect to \tilde{u} . These three equations (2.14)-(2.16) are of the same forms as in the standard 4D FRW solutions algebraically. Therefore, we can follow the procedure given by Elbaz et al. (Elbaz et al. 1997; see also Novello et al. 1996) to derive the canonical formulation of the cosmology.

Firstly, we introduce an expansion parameter $\Theta, \Theta \equiv 3 \frac{\dot{A}}{A}$, which brings (2.14)-(2.16) to

$$\frac{1}{3} \Theta^2 = \rho , \quad (2.17)$$

$$\dot{\Theta}^* + \frac{1}{3} \Theta^2 + \frac{1}{2} (1 + 3\gamma) \rho = 0 , \quad (2.18)$$

$$\dot{\rho}^* + (1 + \gamma) \rho \Theta = 0 . \quad (2.19)$$

Then we choose a new set of canonical variables (q, p) with

$$\begin{aligned} q &= b \rho^{-1/(3+3\gamma)} , \\ p &= \frac{1}{3} b \Theta \rho^{-1/(3+3\gamma)} , \end{aligned} \quad (2.20)$$

where b is an arbitrary constant. In this way, we arrive at the following Hamiltonian

$$\mathcal{H}(q, p) = \frac{1}{2} p^2 - \frac{1}{6} b^{3(1+\gamma)} q^{-(1+3\gamma)} . \quad (2.21)$$

This Hamiltonian describes a particle with momentum p in a potential $V(q) = -\frac{1}{6} b^{3(1+\gamma)} q^{-(1+3\gamma)}$. Using relations (2.17)-(2.21) we can verify that the two Hamilton equations $\dot{q}^* = \partial \mathcal{H} / \partial p$ and $\dot{p}^* = -\partial \mathcal{H} / \partial q$ hold. Be aware that here the time coordinate is not t but \tilde{u} , that is, $\dot{q} \equiv dq/d\tilde{u}$ and $\dot{p} \equiv dp/d\tilde{u}$. This will make a difference between the two canonical formulations of the 4D and 5D cosmologies as we will see in next section.

3 Wave Function and Late-Time Approximation

From the Hamiltonian (2.21) we can employ the standard quantization procedure to write the corresponding Schrödinger equation. Because the time coordinate in the Hamiltonian formulation is \tilde{u} , so the time-dependent wave function is of the form

$$\Psi(q, \tilde{u}) = e^{-i\tilde{E}\tilde{u}}\varphi(q) \quad (3.1)$$

The correspondence principle, $p \rightarrow -i\frac{\partial}{\partial q}$, gives a stationary Schrödinger equation for the stationary wave function $\varphi(q)$ as

$$\frac{1}{2}\frac{\partial^2}{\partial q^2}\varphi(q) + [\tilde{E} + \frac{1}{6}b^{3(1+\gamma)}q^{-(1+3\gamma)}]\varphi(q) = 0. \quad (3.2)$$

In (3.1) and (3.2) \tilde{E} is the associated eigenvalue of the operator $\hat{\mathcal{H}}$, $\tilde{E} = \langle \Psi(q, \tilde{u}) | \hat{\mathcal{H}} | \Psi(q, \tilde{u}) \rangle$. Quantized bound states with negative \tilde{E} of the Schrödinger equation (3.2) were studied (Novello et al. 1996; Mongan 1999, 2000, 2001). Their results can also be used here.

Since \tilde{u} is not the proper time of the cosmic fluid, we are not sure if we can interpret \tilde{E} as the energy. Therefore, we wish to look for the relation between \tilde{u} and the proper time. Now let us consider the 5D metric (2.10), from which we see that the 2D line-element in the $t - y$ plane can be written as

$$ds^2 = -B^2(u)dud(t+y). \quad (3.3)$$

Thus by defining

$$U \equiv \int B^2(u)du, V \equiv t+y. \quad (3.4)$$

we get $ds^2 = -dUdV$. Then let

$$U \equiv T - \lambda Y, V \equiv T + \lambda Y, \quad (3.5)$$

where λ is a constant to be determined later, we get $ds^2 = -dT^2 + \lambda^2 dY^2$. Therefore we find that T is the proper time. Substituting (2.11) in (2.13) and (3.4), we find

$$h\tilde{u} = \frac{2(2+3\gamma)}{3(1+\gamma)}(hu)^{\frac{3(1+\gamma)}{2(2+3\gamma)}} , \quad (3.6)$$

$$hU = (2+3\gamma)(hu)^{\frac{1}{2+3\gamma}} . \quad (3.7)$$

From these two equations we obtain

$$\tilde{u} = W_\gamma U^{\frac{3(1+\gamma)}{2}} = W_\gamma (T - \lambda Y)^{\frac{3(1+\gamma)}{2}} , \quad (3.8)$$

where

$$W_\gamma \equiv \frac{2}{3(1+\gamma)} \left(\frac{h}{2+3\lambda} \right)^{\frac{1+3\gamma}{2}} \quad (3.9)$$

Now let us consider a short period in later times of the Universe. This means that we choose $T = 0$ as the beginning of the Universe (on the $Y = 0$ hypersurface) and write $T = T_0 + \tau$ with $|\tau| \ll T_0$. Then (3.8) gives

$$\begin{aligned} \tilde{u} &= W_\gamma T_0^{\frac{3(1+\gamma)}{2}} \left(1 + \frac{\tau - \lambda Y}{T_0} \right)^{\frac{3(1+\gamma)}{2}} \\ &\approx W_\gamma T_0^{\frac{3(1+\gamma)}{2}} + \frac{3(1+\gamma)}{2} T_0^{\frac{1+3\gamma}{2}} (\tau - \lambda Y) . \end{aligned} \quad (3.10)$$

Using this relation we find that, up to a constant factor, the wave function $\Psi(q, \tilde{u})$ in (3.1) becomes

$$\Psi(q, Y, \tau) \approx e^{-iE\tau} e^{iP_5 Y} \varphi(q) . \quad (3.11)$$

where

$$\begin{aligned} E &\equiv \chi \tilde{E} , \\ P_5 &\equiv \chi \tilde{E} \lambda , \end{aligned} \quad (3.12)$$

and

$$\chi = \frac{3(1+\gamma)W\gamma}{2}T_0^{(1+3\gamma)/2} = \left(\frac{hT_0}{2+3\gamma}\right)^{\frac{1+3\gamma}{2}}. \quad (3.13)$$

Thus we successfully have obtained an approximate wave function $\Psi(q, Y, \tau)$ as shown in (3.11), in which the three variables τ , q and Y were separated.

4 Quantization With a Compact Fifth Dimension

The wave function $\Psi(q, Y, \tau)$ in (3.11) is in a form with three separated variables τ , q and Y . Denote the Y -part of Ψ as $\Phi(Y)$, $\Phi(Y) = e^{iP_5 Y}$, then the momentum operator $\hat{P}_5 = -i\frac{\partial}{\partial Y}$ gives

$$\hat{P}_5 \Phi = P_5 \Phi. \quad (4.1)$$

So P_5 is the eigenvalue of the momentum along the fifth dimension.

Now we suppose the fifth dimension to be a circle with a radius R , i.e., $Y = R\phi$. So we have $\Phi = \Phi(\phi) = e^{iP_5 R\phi}$. The boundary condition requires that $\Phi(\phi)$ must be periodic with period 2π . It follows that we must have

$$P_5 = \frac{n}{R}, \quad n = 0, \pm 1, \pm 2, \dots \quad (4.2)$$

In this way, the fifth momentum P_5 and the corresponding wave function $\Phi(\phi)$ are quantized.

The momentum operator $\hat{P}_5 = -i\frac{\partial}{\partial Y}$, acting on Eq.(4.1) from the left hand side, gives

$$\frac{1}{2}\frac{\partial^2}{\partial Y^2}\Phi(Y) + \frac{n^2}{2R^2}\Phi(Y) = 0. \quad (4.3)$$

Let $\psi(q, Y) \equiv \varphi(q)\Phi(Y)$, then equations (3.2) and (4.3) give

$$\frac{1}{2}\left(\frac{\partial^2}{\partial q^2} + \frac{\partial^2}{\partial Y^2}\right)\psi(q, Y) + \left(\tilde{E} + \frac{n^2}{2R^2} + \frac{1}{6}b^{3(1+\gamma)}q^{-(1+3\gamma)}\right)\psi(q, Y) = 0. \quad (4.4)$$

So we obtain a stationary two-dimensional Schrödinger equation.

5 Conclusion

In this paper we have obtained a Hamiltonian formulation for a $5D$ cosmology. This Hamiltonian leads immediately to a stationary one-dimensional Schrödinger equation (3.2). Thus the corresponding full wave function can describe quantized states of a $5D$ Universe. By using a late-time approximation we have successfully separated the wave function into three parts, $\Psi(q, Y, \tau) \approx e^{-iE\tau} e^{iP_5 Y} \varphi(q)$, corresponding to the proper time τ , the variable q (which describes the radius of the Universe) and the fifth coordinate Y , respectively. Notice that this wave function is just valid in later times of the Universe with $|\tau| \ll T_0$, where T_0 is the age of the Universe at that time. Then by assuming the compact fifth dimension to be a circle, we obtain a quantized spectrum for the momentum P_5 in the fifth direction. Be aware that the q -part of the wave function, $\varphi(q)$, satisfies the one-dimensional Schrödinger equation (3.2), for which bound states were given (Novello et al. 1996). Using their results, as well as the spectrum for P_5 , we can obtain bound states for the whole space-part of the wave function, $\psi(q, Y) = e^{iP_5 Y} \varphi(q)$, which satisfies the two-dimensional Schrödinger equation (4.4). Further studies are needed.

Acknowledgments

This work was supported by NSF of P. R. China under grant 19975007.

References

- Elbaz, E., Novello, M., Salim, J.M., Motta da Silva, M.C. and Klippert, R. (1997). *Hamiltonian Formulation of FRW Equations of Cosmology*, General Relativity and Gravitation, **29**, 481-487.
- Liu, Hongya and Wesson, Paul S. (1994). *Cosmological Solution and their Effective Properties of Matter in Kaluza-Klein Theory*, International Journal of Modern Physics D **3** 627-637.
- Mongan, T.R.(1999), *$N + 1$ dimensional quantum mechanical model for a closed universe*, International Journal of Theoretical Physics **38**, 1521-1529.

- Mongan, T.R.(2000). *Inflation, extra dimensions and Omega sub Lambda approximately =1*, International Journal of Theoretical Physics **39**, 99-104.
- Mongan, T.R.(2001). *A simple Quantum Cosmology*. gr-qc/0103021.
- Novello, M., Salim, J.M., Motta da Silva, M.C., Joras, S.E. and Klippert, R.(1995). *Minimal closed set of observables in the theory of cosmological perturbations*. Phys. Rev. D **51**, 450; *Minimal closed set of observables in the theory of cosmological perturbations. II. Vorticity and gravitational waves*. Phys. Rev. D **52**, 730.
- Novello, M., Salim, J.M., Motta da Silva, M.C. and Klippert, R. (1996). *Canonical formulation of standard cosmology: Direct quantum approach*, Physical Review D **54**, 6202-6205.
- Overduin, J.M., and Wesson, P.S.(1997). *Kaluza-Klein gravity*, Physics Reports, **283**, 303-380.
- Wesson, P.S., Liu, H. and Seahra, S.S.(2000). *The big bang as a higher-dimensional shock wave*, Astronomy and Astrophysics **358**, 425-427.
- Wesson, P.S., 1999, *Space, Time, Matter* (Singapore: World Scientific).
- Wheeler, J.A., DeWitt, M.C., and DeWitt, B.S. (1964). *In Relativity, Groups and Topology*, DeWitt, B., DeWitt, C., eds. (Gordon Breach, New York).
- Anchordoqui, L., Nunez, C., and Olsen, K.(2000). *Quantum Cosmology and AdS/CFT*, JHEP **0100**, 050.